1. Convex Algebraic Geometry Exercises

Exercise 1.1. For a polynomial p define its *Newton Polytope* $\mathcal{N}(p)$ to be the convex hull of the vectors of exponents of monomials that occur in p. For example, if $p = x_1x_2^2 + x_2^2 + x_1x_2x_3$ then $\mathcal{N}(p) = \text{conv}(\{(1,2,0),(0,2,0),(1,1,1)\})$, which is a triangle in $\mathbb{R}^3$.

Show that if $p = \sum q_i^2$ then

$$\mathcal{N}(q_i) \subseteq \frac{1}{2}\mathcal{N}(p).$$

Exercise 1.2. Let $M(x,y,z)$ be the Motzkin form:

$$M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2.$$

(a) Show that M is a nonnegative form and find all of the zeroes of M.

(b) Calculate the Newton Polytope of the Motzkin form and use Exercise 1.1 to show that the Motzkin form is not a sum of squares.

Exercise 1.3. Let S be the real variety defined by the polynomial $f = (x^2+y^2)^2 - x^2 + y^2$:

$$S = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}.$$

(a) Find the minimal $c \in \mathbb{R}$ such that the polynomial $c - y$ is nonnegative on S.

(b) For the value of c from part (a), find a sums of squares certificate for nonnegativity of $c - y$ on S.

Exercise 1.4. Let $K \subset \mathbb{R}^n$ be a compact convex set with the origin not in K. Show that the conical hull of K, $\text{cone}(K)$ is closed. Construct an explicit example that shows that the condition $0 \notin K$ is necessary.

Exercise 1.5. Let $\mathbb{R}[x]_{n,d}$ be the vector space of real polynomials in n variables of degree at most d.

(a) Let $P_{n,2d}$ be the set of nonnegative polynomials in $\mathbb{R}[x]_{n,2d}$:

$$P_{n,2d} = \{p \in \mathbb{R}[x]_{n,2d} \mid p(x) \geq 0 \text{ for all } x \in \mathbb{R}^n\}.$$

Show that $P_{n,2d}$ is a closed full-dimensional convex cone in $\mathbb{R}[x]_{n,2d}$.

(b) Let $\Sigma_{n,2d}$ be the set sums of squares in $\mathbb{R}[x]_{n,2d}$:

$$\Sigma_{n,2d} = \left\{p \in \mathbb{R}[x]_{n,2d} \mid p(x) = \sum q_i^2 \text{ for some } q_i \in \mathbb{R}[x]_{n,d}\right\}.$$

Show that $\Sigma_{n,2d}$ is a closed full-dimensional convex cone in $\mathbb{R}[x]_{n,2d}$. (Hint: Use Exercise 1.4)

Exercise 1.6. (from Bernd Sturmfels) Determine the maximum number of edges that can appear in the convex hull of a compact curve of degree 4 in the plane. How about degree 6?

2. Tropical Exercises

(1) Newton polytopes and Minkowski sums. See Section 2.3 of the tropical book draft: `http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.pdf` for definitions.

  (a) Show that the vertices of the Minkowski sum of two polytopes P_1 and P_2 have the form $v_1 + v_2$ where v_1 and v_2 are vertices of P_1 and P_2 respectively.

  (b) Show that any face F of the Minkowski sum of two polytopes P_1 and P_2 can be written uniquely as $F = F_1 + F_2$ where F_1 and F_2 are faces of P_1 and P_2 respectively.

  (c) Show that the Newton polytope of the product fg of two polynomials f and g is the Minkowski sum of the Newton polytopes of f and g. What can you say about multiplicities?

  (d) Show that the normal fan of the Minkowski sum P_1+P_2 is the common refinement of the normal fans of P_1 and P_2.

  (e) Show the tropical hypersurface $\mathcal{T}(fg)$ of the product fg is the union of the tropical hypersurfaces $\mathcal{T}(f)$ and $\mathcal{T}(g)$.

  (f) Describe the Newton polytope of the polynomial

$$(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$

  Do you know its name? Describe the tropical hypersurface of the polytope.

(2) Non-constant coefficient tropical hypersurfaces.

  (a) Draw the Newton polytope and tropical hypersurface of the tropical polynomial

$$f = 1 \oplus x \oplus (1 \odot x^2) \oplus y \oplus (x \odot y) \oplus 1 \odot y^2.$$

  (b) Consider the "lifted" Newton polytope of f: the convex hull of the points (a, b, c) where $a \odot x^b \odot y^c$ is a monomial in f. Projection of its *lower faces* (those whose inward-point normal vectors have positive first coordinate) gives a *regular subdivision* of the Newton polytope of f. Draw this subdivision.

  (c) What is the relationship between the subdivision and the tropical hypersurface?

(3) Mixed volumes and Bernstein's Theorem.
Consider two polynomials in $\mathbb{C}[x, y]$ with the support sets $\{1, x^2y, xy^2\}$ and $\{1, x, x^2, x^2y\}$ respectively with generic coefficients. What is the number of their common zeroes in $(\mathbb{C}^*)^2$? Choose a lifting, draw the mixed subdivision, and compute the mixed volume.

3. Computational Algebraic Geometry Exercises

SOLVING QUADRATIC EQUATIONS

Exercise 3.1. (a) Show that solving every (0-dimensional) system of equations $F(\boldsymbol{x}) = 0$ is *equivalent* to solving a system of quadratic equations $G(\boldsymbol{y}) = 0$ in the following sense: there is a map $\phi : k[\boldsymbol{y}] \to k[\boldsymbol{x}]$ such that ϕ is an isomorphism of $\mathbb{V}(F)$ and $\mathbb{V}(G)$ and the latter is defined by

$$G(\boldsymbol{y}) = F(\phi(\boldsymbol{y})) \cup \{y_i - \phi(y_i) : i \in [n]\},$$

which are polynomials of degree 2.

(b) Construct such ϕ for

$$\begin{aligned} F = \{&4x_2x_3^8 - 5x_1x_3^3 - 3x_1^2x_2 + x_1x_2^2 - 8, \\ &x_3^9 - 3x_1x_2x_3^5 + x_1x_3^3 - 7x_1^2x_2 - 2x_1x_2^2 - x_2 - 1, \\ &2x_1x_2x_3^9 + 5x_1x_3^9 + 5x_2x_3^8 - x_1^2x_2 - 4x_1x_2^2 + 5x_3 + 1\} \end{aligned}$$

(c) (!!!experiment!!!) Try computing the Gröbner basis of $\langle G \rangle$, eliminate all but one variables, compute numerically the points of $\mathbb{V}(G)$. How does the performance of these compare to that of the same procedures executed for $\langle F \rangle$ and $\mathbb{V}(F)$?

(d) What is the Waring rank of a (quadratic) homogeneous polynomial? Prove that any projective variety (in $\mathbb{P}^n$) is isomorphic to a projective variety (embedded in $\mathbb{P}^N$ for some N) cut out by a linear polynomial and quadtratic polynomials of rank at most 4.

Exercise 3.2. Consider a homotopy

$$H(x,t) = (1-t)G(\boldsymbol{x}) + \gamma t F(x).$$

Let $G = x^2 - 1$. Find all $\gamma \in \mathbb{C}$ such that there exists $t \in \mathbb{R}$ with a singular solution x to $H(x,t) = 0$

(a) for $F = x^2 + 2x - 3$;

(b) for $F = x^2 + ax + b$ ($a, b \in \mathbb{R}$).

(c) What is the real dimension of the set of such γ?

Exercise 3.3. (a) Find an approximate zero $\tilde{x} \in \mathbb{Q}$ (in the sense of Smale's α-theory) associated to the positive root of $ax^2 - 1 = 0$, $a \in \mathbb{N}$.

(b) How does the number of digits needed to write down your $\tilde{x}$ depend on a?