On the Stability of Solutions in Numerical Algebraic Geometry

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ATLANTA
A double Fibration Scheme (inspired in Shub & Smale)
First details

Output Space "O"

Solution Variety "V"

Input Space "I"

Usually $\mathbb{R}^n$ or $\mathbb{C}^n$

$\mathbb{P}(\mathbb{R}^n)$ or $\mathbb{P}(\mathbb{C}^n)$

Other manifold (Grassmannian...)

Usually defined by $ev^{-1}(0)$,

$ev: I \times O \rightarrow O^*$

Usually $\mathbb{R}^n$ or $\mathbb{C}^n$

$\mathbb{P}(\mathbb{R}^n)$ or $\mathbb{P}(\mathbb{C}^n)$

Other manifold or parametrized set
Some examples

Output Space "O"

Solution Variety "V"

PZF:
\[ I = \{ \text{polynomials, degree } \leq d \} \]
\[ \Theta = \mathbb{C} \]
\[ \psi : I \times \Theta \to \mathbb{C} \]
\[ (p, z) \mapsto p(z) \]

EVP:
\[ I = \mathbb{C}^{n \times n} \]
\[ \Theta = \mathbb{C} \]
\[ \varphi : I \times \Theta \to \mathbb{C} \]
\[ (A, z) \mapsto \lambda I - A \]

MIMO:
\[ I = \{ \text{Matrices } H \in \mathbb{C}^{k \times k} \} \]
\[ \Theta = \{ \text{Grassmannians } U_k, V_k \} \]
\[ \psi (H, U, V) = (U^T H U V)_{k \times k} \]

Input Space "I"

PSS:
\[ I = \{ \text{pol. systems with } \leq n \text{ eqns.} \}
\[ \Theta = \mathbb{P}(\mathbb{C}^n) \]
\[ \psi (f, \lambda) = \psi (f) (\text{defined in bundles}) \]
The condition number

\[ \frac{\|\Theta\|}{\|j\|} \|\| \]  
Relative change in input

\[ \mu(i, \Theta) \]  
Condition number

Requires "norms" in \( I, O \)
**Metrics**

Output Space "O"

- Usually has a natural metric

Solution Variety "V"

- Metric inherited from the product

Input Space "I"

- Natural metrics:
  - \( \sup_{\theta \in \Theta} \| \varphi - (i, \theta) \| \)
  - \( \left[ \int_{\Theta} \| \varphi - (i, \theta) \|^2 \, d\theta \right]^{1/2} \)
  - If \( I = \mathbb{C}^n \) or \( \mathbb{R}^n \rightarrow 2\)-norm
Homotopy Method for PSS

Output Space "O"

Solution Variety "V"

Input Space "I"

Number of "O" \leq \text{length in \(\mu\)-metric}

Constructive versions:
- Bürgisser - Cucker (almost)
- B.- Leluyer (Q-arithmetic)
- Dedieu - Malajovich - Saub

Similar for EVP by ABBCS
Open problems: PEVP, MIMO, Structured PSS

Homotopy method

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Condition metric and complexity

Output Space "O"

Solution Variety "V"

A new metric here:
\( \mu(i,o) \cdot \text{(product metric)} \)

"The condition metric"

Shub: Complexity of
PSS Homotopy
is at most C

length in the condition metric

Input Space "I"

Poly-time for EVP

Ansultano-Burgisser-Cucker
- Shub:

Same for EVP

Solution to Swale B.
Burgisser-Cucker
Lairez
Smale’s $\alpha$–theory for polynomial systems

There is a constant $\alpha_0 \geq 0.15$ and there are quantities

$$\gamma(f, z) = \sup_{k \geq 2} \left\| f^{(k)}(z)/(k!f'(z)) \right\|^{1/(k-1)}$$

$$\beta(f, z) = |f(z)/f'(z)|$$

$$\alpha(f, z) = \beta(f, \zeta) \gamma(f, z)$$

such that $\alpha(f, z) \leq \alpha_0$ implies that Newton’s method with starting point $z$ converges to some zero $\zeta$ of $f$.

Practical version and implementation by Hauenstein and Sottile’s $\text{AlphaCercified}$.

Version with $\mu$ (Shub and Smale): if $\text{distance}(z, \zeta) \leq c/\mu(f, \zeta)$ then there is convergence.
The condition number theorem

Output Space "Ω"

Solution Variety "V"

\[
\text{distance } \frac{1}{\mu(i, Ω)} \text{ points where } \mathcal{N}_i \text{ has no local inverse}
\]

Holds for:
- PSS (Shub & Smale)
- EVP (Armentano - Cucker, Armentano, ...)
- PEVP (De dies & Tisseur)
- Sparse PSS (Kalajovich)

In a general setting: Denzuel

Input Space "I"

Historical result:
- Schmidt, Mirsky, Eckart, Young
  for \( Ax = b \)
Convexity properties of the condition number

Output Space "Ω"

Solution Variety "V"

Input Space "I"

Shortest path?

Geometric properties

Known in the PSS case

B. Shub

B. Redieu, Malajovich, Shub

Juan G. Guadalupe del Rey

P EVP, EVP, Span...?
A machine for computing integrals

Double fibration & Integrals

Output Space "Ω"

Solution Variety "V"

Integral in I transform into integrals in Ω

Key: Federer’s coarea formula

\[
\int F(i, o) \nu \left( N_{\Pi_1} (i, o) \right) \, d\nu
\]

This can be done if \( \Pi_1, \Pi_2 \) are almost submersions.

\[
\sum_{i \in I} F(i, o) \, d\nu
\]

\[
\int_{(i, o) \in V} \frac{F(i, o)}{N_{\Pi_1} (i, o)} \, d\nu
\]
Expected Condition Number for Linear Algebra

Theorem (Smale, Demmel, Edelman, Sutton, Chen, Dongarra)

Let $A \in \mathbb{C}^{n \times n}$ be chosen at random with independent entries following the standard complex Gaussian distribution $\mathcal{N}_\mathbb{C}(0, 1)$. Then, the expected value of the condition number ($\|A\| \|A^{-1}\|$) for Linear Algebra satisfies:

$$E_A(\kappa(A)) \leq 40n^2.$$
Theorem (Shub, Smale, B., Pardo)

Let $f$ be a homogeneous system ($n + 1$ variables, $n$ equations, degrees $d_1, \ldots, d_n$) chosen at random with independent entries following the so-called Bombieri distribution. Then, the expected value of the squared condition number for PSS satisfies

$$E_f \left( \frac{1}{D} \sum_{\zeta \text{ a zero of } f} \mu(f, \zeta)^2 \right) = N \left( n \left( 1 + \frac{1}{n} \right)^n - 2n - 1 \right) \leq nN.$$

Here, $D = d_1 \cdots d_n$ is the generic number of solutions and $N$ is the number of monomials in dense encoding.

See also smoothed analysis by Burgisser and Cucker.
Theorem (Armentano, B., Bürgisser, Cucker, Shub)

Let $A \in \mathbb{C}^{n \times n}$ be chosen at random with independent entries following the standard complex Gaussian distribution $\mathcal{N}_\mathbb{C}(0,1)$. Then, the expected value of the condition number for eigenvector computation satisfies:

$$E_A \left( \frac{1}{n} \sum_{(\lambda, \nu) \text{ eigenpair for } A} \mu(A, \lambda, \nu)^2 \right) \leq n^3.$$ 

See P. Breiding’s paper for a related problem.
The Polynomial Eigenvalue Problem (PEVP)

Output Space "Θ" = \mathbb{P}(\mathbb{C}^2)

Solution Variety "V"

\[ \sigma(A, (\alpha, \beta)) = \left| \sum_{k=0}^{d} \alpha^{k} \beta^{-k} A_{k} \right| \]

\( \mu(A, (\alpha, \beta)) \) has a known formula

Example useful in apps: QEVP
\[ |A_0 \beta^2 + A_1 \alpha \beta + A_2 \alpha^2| = 0 \]

Input Space "I" = \{ (A_0, ..., A_d) | n \times n matrices \}

See works by Mehrmann-Voss, Tisseur, Dedieu-Tisseur - Meerbergen

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Theorem (Armentano & B., 2017)

Let $A = (A_0, \ldots, A_d) \in (\mathbb{C}^{n \times n})^{d+1}$ be chosen at random with independent entries following the standard complex Gaussian distribution $\mathcal{N}_\mathbb{C}(0, 1)$. Then, the expected value of the squared condition number for the PEVP satisfies:

$$E_A \left( \frac{1}{dn} \sum_{(\alpha, \beta) \in \text{Eig}(A)} \mu(A, (\alpha, \beta))^2 \right) = \left( d + 1 \right) \frac{n^2 - 1}{d}.$$

Note: this result can be adapted to sparse situations.
REFERENCES
Disclaimer: the list is by no means complete!
Selected references for the material in this talk

The condition number of polynomial systems


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Selected references for the material in this talk
On the theoretical complexity of solving polynomial systems


Selected references for the material in this talk

On the theoretical complexity of solving polynomial systems


- D. Armentano and M. Shub, “Smale’s Fundamental Theorem of Algebra Reconsidered”, J. FoCM 14, 2014. (See also the Erratum in the same journal).

Selected references for the material in this talk

On the geometrical aspects and the condition metric

• C. Beltrán and M. Shub, Complexity of Bezout’s Theorem VII: Distance Estimates in the Condition Metric., J. FoCM, 9 (2009), no. 2, 179–195.


• C. Beltrán, Jean Pierre Dedieu, Gregorio Malajovich, Michael Shub. ”Convexity properties of the condition number II”. SIAM Journal of Matrix Analysis and Applications 33, no. 3 (2012), pp. 905-939

Selected references for the material in this talk
Explicit algorithms for PSS and software

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Explicit algorithms for PSS and software

- J. Verschelde. *PHCpack: a general-purpose solver for polynomial systems by homotopy continuation*
- J. Hauenstein and F. Sottile. *Alpha–certified* [website](http://www.math.tamu.edu/~sottile/research/stories/alphaCertified/)
Selected references for the material in this talk

About the condition number in different problems

Selected references for the material in this talk

The condition number of eigenvalue problems


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Selected references for the material in this talk
Surveys and books with a lot of this material